SAMPLE MPCS Mathematics Placement Test

This is a sample exam of practice problems to help you prepare for the MPCS math placement exam. An answer key is included.

Problems similar to the practice problems will be on the actual exam. You will have 2 hours to complete the actual exam. During the exam you will not be permitted to use books, notes, phones or any other electronic devices.

You should expect the problems on the actual exam to consist of mostly proof-based problems, with some computational problems. You should answer 70% of the problems correctly in order to pass the exam. Skipping any one of the major topics on the Topics List will likely result in your failing the exam.
Problem 1.

1(a) [2 points]
Compute $61^{61} \pmod{9}$ by modular arithmetic.

1(b) [1 points]
State Fermat's Little Theorem.

1(c) [2 points]
Use part (b) and modular arithmetic to compute $61^{61} \pmod{11}$.

Problem 2.
Let $p$, $q$, and $r$ be distinct prime numbers.

2(a) [3 points]
How many integers $n$ with $1 \leq n \leq pqr$ are relatively prime to $pqr$?

2(b) [2 points]
Let $p = 2$, $q = 3$, and $r = 5$. Write down all integers $n$ with $1 \leq n \leq pqr$ and relatively prime to $pqr$, and show that your formula from part a gives the correct number.

Problem 3.
Find the multiplicative inverse of $a \pmod{m}$ for each of the following pairs of integers or prove that it does not exist:

3(a) [1 points]
$a = 15$, $b = 72$.

3(b) [1 points]
$a = 55$, $b = 204$.

3(c) [3 points]
$a = k^2 - 1$, $b = k + 1$. 
Problem 4.
Use the Chinese Remainder Theorem to find all solutions of the following system of congruences.

\[
\begin{align*}
  x &\equiv 1 \pmod{2} \\
  x &\equiv 2 \pmod{3} \\
  x &\equiv 3 \pmod{5} \\
  x &\equiv 4 \pmod{11}
\end{align*}
\]

Problem 5.
How many ways can \( n \) books be placed on \( k \) distinguishable shelves:

5(a) [2 points]
If the books are indistinguishable copies of the same title?

5(b) [3 points]
If no two books are the same and the positions of the books on the shelves matter?

Problem 6.
You are organizing a dance with students attending from 4 departments: there are 4 CS students, 4 math students, 4 physics students, and 6 statistics students. For one of the dances, you need to arrange 10 students to stand in a circle, all facing the center and holding hands.

6(a) [2 points]
How many different circles of students can you arrange? Two circles are the same if and only if everyone is holding hands with the same neighbors using the same hands. Do not simply your answer (e.g., you can leave in factorials, etc.)

6(b) [3 points]
How many different circles of students can you arrange if there must be exactly 1 math student and 1 physics student, with the math student to the right of the physics student?
Problem 7.

7(a) [2 points]
Prove: if 7 integers are selected from the first 10 positive integers, there must be at least two pairs of these integers with the sum 11. Is this statement true if 6 integers are selected rather than 7?

7(b) [3 points]
Prove: whenever 25 girls and 25 boys are seated around a circular table, there is always a person both of whose neighbors are boys.

Problem 8.

Let $S$ be the sample space of two-digit odd integers (with leading zeros allowed) $n = d_2d_1$ with $01 \leq n \leq 99$ chosen at random with equal probabilities.

8(a) [1 points]
What is the size of the sample space $S$?

8(b) [1 points]
Find the probability of the event $A$ that an element of $S$ has distinct digits $d_1 \neq d_2$.

8(c) [1 points]
Define a random variable $R$ to be the sum of the digits of $n$: thus $R(n) = d_1 + d_2$. Find $E(R)$, the expected value of $R$.

8(d) [1 points]
Let $B$ be the event that $R(n) = 10$ and event $A$ as in part b. Find the conditional probability $p(B|A)$.

8(e) [1 points]
Let $C$ be the event that $d_2 \geq 5$ and $D$ be the event that $d_1 \leq 5$. Are $C$ and $D$ independent?
Problem 9.

9(a) [1 points]
Let \( S \) be a sample space. Complete the definition: “A random variable on \( S \) is ...

9(b) [1 points]
Prove: If \( X_i, i = 1, 2, \ldots, n \), \( n \) a positive integer, are random variables on \( S \), then \( E(X_1 + X_2 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n) \).

9(c) [2 points]
Suppose that \( X \) and \( Y \) are random variables on \( S \) and that \( X \) and \( Y \) are nonnegative for all elements \( s \) in \( S \). Let \( U \) be the random variable defined by \( U(s) = \max(X(s), Y(s)) \) for all \( s \) in \( S \). Prove that \( E(U) \leq E(X) + E(Y) \).

9(d) [2 points]
Suppose that \( X(s) \leq M \), where \( M \) is a positive real number, for all \( s \) in \( S \). Let \( Z(s) = X(s) \cdot Y(s) \). Prove that \( E(Z) \leq M \cdot E(Y) \).

Problem 10.
Let \( S \) be the set of permutations of 10 objects.

10(a) [2 points]
How many elements of \( S \) leave the third object fixed?

10(b) [3 points]
Consider the experiment of choosing an element of \( S \) at random with equal probability. Let \( R \) be the random variable defined by \( R(\sigma) = |\{j : \sigma(j) = j\}| \) where \( \sigma \) is a permutation in \( S \), so \( R(\sigma) \) is the number of fixed points of \( \sigma \). Compute the expected value of \( R \).

Problem 11.
Suppose that two fair 8-sided dice are rolled.

11(a) [2 points]
What is the expected value of the sum of the numbers that come up?

10(b) [3 points]
What is the variance of the sum of the numbers that come up?
Problem 12.
Is the following pair of graphs isomorphic? If yes, give an explicit isomorphism; if no, explain why they are not isomorphic.

![Graphs G and H](image)

Problem 13.

13(a) [3 points]
Prove that every connected undirected simple graph on $n$ vertices has at least $n - 1$ edges.

13(b) [2 points]
Prove that every undirected simple graph on $n$ vertices with $k$ connected components has at least $n - k$ edges.

Problem 14.

14(a) [1 points]
Complete the following definition: “A tree is . . .”.

14(b) [4 points]
Prove that a simple undirected graph is a tree if and only if it contains no simple circuits and the addition of an edge connecting two nonadjacent vertices produces a new graph that has exactly one simple circuit (where circuits that contain the same edges are not considered different).
Problem 15.
Let $G$ be a nonempty simple undirected graph.
Prove: if every vertex of $G$ has even degree, then $G$ contains a simple circuit.

Problem 16.
Find a recurrence to describe the number of ways to completely cover a $2 \times n$ checkerboard with $1 \times 2$ dominoes.
Problem 1.

1(a) [2 points]
Compute $61^{61} \pmod{9}$ by modular arithmetic.

Ans. Since $61 \equiv -2 \pmod{9}$, we have

$$61^{61} \equiv -2^{61} \equiv -2 \cdot 8^{60} \equiv -2 \cdot (-1)^{61} \equiv -2 \equiv 7 \pmod{9}.$$

1(b) [1 points]
State Fermat’s Little Theorem.

Ans. If $p$ is prime, then for any integer $a$, $a^p \equiv a \pmod{p}$. If $\gcd(a, p) = 1$, then $a^{p-1} \equiv 1 \pmod{p}$.

1(c) [2 points]
Use part (b) and modular arithmetic to compute $61^{61} \pmod{11}$.

Ans. Since $p = 11$ is prime and $\gcd(61, 11) = 1$, Fermat’s Little Theorem implies that $61^{10} \equiv 1 \pmod{11}$. Hence $61^{61} \pmod{11} \equiv 61 \equiv 6 \pmod{11}$. 

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Problem 2.
Let \( p, q, \) and \( r \) be distinct prime numbers.

2(a) [3 points]
How many integers \( n \) with \( 1 \leq n \leq pqr \) are relatively prime to \( pqr \)?

**Ans.** Define
\[
P = \{1 \leq n \leq pqr \mid p|n\}
\]
and similarly for \( Q \) and \( R \). Then \( |P| = pqr/p = qr \), while \( |Q| = pr \) and \( |R| = pq \). Similarly \( |P \cap Q| = pqr/pq = r \) since \( p, q, \) and \( r \) are all distinct primes, and \( |P \cap R| = q, |Q \cap R| = p \). Finally, \( |P \cap Q \cap R| = 1 \), so we can apply inclusion-exclusion:

\[
|\{1 \leq n \leq pqr \mid \gcd(n, pqr) = 1\}| = pqr - |\{1 \leq n \leq pqr \mid p|n \lor q|n \lor r|n\}| \\
= pqr - |P \cup Q \cup R| \\
= pqr - qr - pr - pq + r + q + p - 1.
\]

2(b) [2 points]
Let \( p = 2, q = 3, \) and \( r = 5 \). Write down all integers \( n \) with \( 1 \leq n \leq pqr \) and relatively prime to \( pqr \), and show that your formula from part b gives the correct number.

**Ans.** Putting \( p = 2, q = 3, r = 5 \) gives eight numbers 1, 7, 11, 13, 17, 19, 23, 29 between 1 and \( pqr = 30 \) inclusive and relatively prime to 30. The formula from part b gives

\[
pqr - qr - pr - pq + r + q + p - 1 = 30 - 15 - 10 - 6 + 5 + 3 + 2 - 1 = 8.
\]
Problem 3.
Find the multiplicative inverse of $a \pmod{m}$ for each of the following pairs of integers or prove that it does not exist:

3(a) [1 points]
$a = 15, b = 72.$

Ans. Does not exist because $\gcd(15, 72) = 3 \neq 1.$

3(b) [1 points]
$a = 55, b = 204.$

Ans. We use the Euclidean algorithm to find that $\gcd(55, 204) = 1$ and to write as a linear combination of 55 and 204:

\[
\begin{align*}
204 & = 3 \cdot 55 + 39 \quad (\Rightarrow 39 = -3 \cdot 55 + 204) \\
55 & = 1 \cdot 39 + 16 \quad (\Rightarrow 16 = 55 - 39 = 4 \cdot 55 - 204) \\
39 & = 2 \cdot 16 + 7 \quad (\Rightarrow 7 = 39 - 2 \cdot 16 = -11 \cdot 55 + 3 \cdot 204) \\
16 & = 2 \cdot 7 + 2 \quad (\Rightarrow 2 = 16 - 2 \cdot 7 = 26 \cdot 55 - 7 \cdot 204) \\
7 & = 3 \cdot 2 + 1 \quad (\Rightarrow 1 = 7 - 3 \cdot 2 = -89 \cdot 55 + 24 \cdot 204)
\end{align*}
\]

Now $1 = -89 \cdot 55 + 24 \cdot 204 \equiv -89 \cdot 55 \pmod{204}$ so the multiplicative inverse of $55 \equiv -89 \equiv 115 \pmod{204}.$

3(c) [3 points]
$a = k^2 - 1, b = k + 1.$

Ans. Does not exist because $\gcd(k^2 - 1, k + 1) = k + 1 \neq 1.$
Problem 4.

Use the Chinese Remainder Theorem to find all solutions of the following system of congruences.

\[
\begin{align*}
    x &\equiv 1 \pmod{2} \\
    x &\equiv 2 \pmod{3} \\
    x &\equiv 3 \pmod{5} \\
    x &\equiv 4 \pmod{11}
\end{align*}
\]

Ans. The Chinese remainder theorem needs to be used here. Following the notation on page 278 of Rosen, \(a_1 = 1, m_1 = 2, a_2 = 2, m_2 = 3, a_3 = 3, m_3 = 5, a_4 = 4, m_4 = 11, m = 330, M_1 = 330/2 = 165, M_2 = 330/3 = 110, M_3 = 330/5 = 66, M_4 = 330/11 = 30\). We find inverses \(y_i\) of \(M_i\) modulo \(m_i\) for \(i = 1, 2, 3, 4\): \(y_1 = 1, y_2 = 2, y_3 = 1,\) and \(y_4 = 7\), respectively (for the last inverse, \(30 \equiv 8 \pmod{11}\), so we want to solve \(8y_4 \equiv 1 \pmod{11}\), i.e., \(y_4 = 7\)). The simultaneous solution is \(x = 1 \cdot 165 \cdot 1 + 2 \cdot 110 \cdot 2 + 3 \cdot 66 \cdot 1 + 4 \cdot 30 \cdot 7 = 1643 \equiv 323 \pmod{330}\). Thus the solutions are all integers of the form \(323 + 330k\), where \(k\) is an integer.
Problem 5.
How many ways can \( n \) books be placed on \( k \) distinguishable shelves:

5(a) [2 points]
If the books are indistinguishable copies of the same title?

Ans. This is a “stars and bars” problem. Let \( x_i \) be the number of copies of the book placed on shelf \( i \). The answer is the number of solutions to the equation \( x_1 + x_2 + \cdots + x_k = n \), where each \( x_i \) is a nonnegative integer: \( \binom{n+k-1}{k-1} \).

5(b) [3 points]
If no two books are the same and the positions of the books on the shelves matter?

Ans. First choose the locations for the books, which was counted in part (a), and then choose a permutation of the \( n \) books to put into those locations shelf by shelf, from the top down, and from left to right on each shelf. Thus the answer is \( \binom{n+k-1}{k-1} n! \).
Problem 6.
You are organizing a dance with students attending from 4 departments: there are 4 CS students, 4 math students, 4 physics students, and 6 statistics students. For one of the dances, you need to arrange 10 students to stand in a circle, all facing the center and holding hands.

6(a) [2 points]
How many different circles of students can you arrange? Two circles are the same if and only if everyone is holding hands with the same neighbors using the same hands. Do not simply your answer (e.g., you can leave in factorials, etc.)

Ans. There are 18 students altogether. There are \( \binom{18}{10} = \frac{18!}{8!} \) ways to arrange 10 of the students in different orders standing in a row (not a circle). These orders may be grouped into groups of 10, corresponding to the same circle, depending of where the first person in the circle is, and proceeding clockwise. So the answer is \( \binom{18}{10}/10 \).

6(b) [3 points]
How many different circles of students can you arrange if there must be exactly 1 math student and 1 physics student, with the math student to the right of the physics student?

Ans. Proceeding clockwise around the circle, we must encounter the math student and physics student in the order M-P. The number of ways to assign the M and the P is \( 4 \times 4 = 16 \). The number of spaces continuing clockwise from P around to M is 8. There are \( P(10, 8) \) ways to order 8 of the remaining 4 CS students and 6 statistics students to fit into these spaces. Thus there are \( 16 \times P(10, 8) = 16 \times 10!/2! \) circles.
Problem 7.

7(a) [2 points]
Prove: if 7 integers are selected from the first 10 positive integers, there must be at least two pairs of these integers with the sum 11. Is this statement true if 6 integers are selected rather than 7?

Ans. Group the first 10 positive integers into five subsets of two integers each, each subset adding up to 11: \{1, 10\}, \{2, 9\}, \{3, 8\}, \{4, 7\}, \{5, 6\}. If we select 7 integers from this set, then by the pigeonhole principle (at least) two of them must come from the same subset. These two integers have a sum of 11. Now if we ignore the two in the same subset, there are 5 more integers and 4 more subsets; again by the pigeonhole principle (at least) two of them must come from the same subset. This gives us two pairs of integers, as desired. The statement is not true if 6 integers are selected: the set \{1, 2, 3, 4, 5, 6\} has only 5, 6 from the same subset, so there is only one pair with a sum of 11.

7(b) [3 points]
Prove: whenever 25 girls and 25 boys are seated around a circular table, there is always a person both of whose neighbors are boys.

Ans. Number the seats around the table from 1 to 50, with seat 50 adjacent to seat 1. There are 25 seats with odd numbers and 25 seats with even numbers. If no more than 12 boys occupied the odd-numbered seats, then at least 13 boys would occupy the even-numbered seats, and vice versa. Assume that at least 13 boys occupy the 25 odd-numbered seats. Then at least two of those boys must be in consecutive odd-numbered seats. The person sitting between those two boys will have boys as both of his or her neighbors.
Problem 8.
Let $S$ be the sample space of two-digit odd integers (with leading zeros allowed) $n = d_2d_1$ with $01 \leq n \leq 99$ chosen at random with equal probabilities.

8(a) [1 points]
What is the size of the sample space $S$?

Ans. 50.

8(b) [1 points]
Find the probability of the event $A$ that an element of $S$ has distinct digits $d_1 \neq d_2$.

Ans. Since $A$ is all of $S$ except for 11, 33, 55, 77, and 99, $A$ has cardinality 45 and $p(A) = 45/50 = 9/10$.

8(c) [2 points]
Define a random variable $R$ to be the sum of the digits of $n$: thus $R(n) = d_1 + d_2$. Find the expected value $E(R)$.

Ans. Since expectation is linear, $E(R) = E(d_2) + E(d_1)$. Since the digits are equally probable, $E(d_2) = (0+1+2+\cdots+9)/10$ and $E(d_1) = (1+3+5+7+9)/5$. Summing, $E(R) = 9.5$.

8(d) [2 points]
Let $B$ be the event that $R(n) = 10$ and $A$ be as in part b. Find the conditional probability $p(B|A)$.

Ans. If $R(n) = 10$, then $n$ is one of the 5 numbers 19, 37, 55, 73, and 91. Only 55 is not in $A$, so

$$p(B|A) = \frac{p(B \cap A)}{p(A)} = \frac{4/50}{45/50} = \frac{4}{45}.$$ 

8(e) [1 points]
Let $C$ be the event that $d_2 \geq 5$ and $D$ be the event that $d_1 \leq 5$. Are $C$ and $D$ independent?

Ans. Yes: $p(C \cap D) = p(C) \cdot p(D)$. Proof: $p(C \cap D) = 15/50$, $p(C) = 25/50$, $p(D) = 30/50$, so

$$p(C) \cdot p(D) = \frac{30}{50} \cdot \frac{25}{50} = \frac{15}{50} = p(C \cap D).$$
Problem 9.

9(a) [1 points]
Let $S$ be a sample space. Complete the definition: “A random variable on $S$ is . . .”.

Ans. A random variable on $S$ is a function that assigns a real number to each element $s \in S$.

9(b) [1 points]
Prove: If $X_i$, $i = 1, 2, \ldots, n$, $n$ a positive integer, are random variables on $S$, then $E(X_1 + X_2 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n)$.

Ans.

\[
E(X_1 + X_2) = \sum_{s \in S} p(s)(X_1(s) + X_2(s))
\]

\[
= \sum_{s \in S} p(s)X_1(s) + \sum_{s \in S} p(s)X_2(s)
\]

\[
= E(X_1) + E(X_2)
\]

Prove true for $n > 2$ by induction.

9(c) [2 points]
Suppose that $X$ and $Y$ are random variables on $S$ and that $X$ and $Y$ are non-negative for all elements $s$ in $S$. Let $U$ be the random variable defined by $U(s) = \max (X(s), Y(s))$ for all $s \in S$. Prove that $E(U) \leq E(X) + E(Y)$.

Ans. Since $X$ and $Y$ are nonnegative on $S$, $X(s) \leq X(s) + Y(s)$ and $Y(s) \leq X(s) + Y(s)$ for all $s \in S$. Since $U(s)$ is either $X(s)$ or $Y(s)$ for all $s \in S$, $U(s) \leq X(s) + Y(s)$ for all $s \in S$. Therefore $E(U) \leq E(X+Y) = E(X) + E(Y)$.

9(d) [2 points]
Suppose that $X(s) \leq M$, where $M$ is a positive real number, for all $s \in S$. Let $Z(s) = X(s) \cdot Y(s)$. Prove that $E(Z) \leq M \cdot E(Y)$.

Ans.

\[
E(Z) = \sum_{s \in S} p(s) \cdot Z(s) \quad \text{(by definition)}
\]

\[
= \sum_{s \in S} p(s)(X(s) \cdot Y(s))
\]

\[
\leq \sum_{s \in S} p(s)(M \cdot Y(s)) \quad \text{(since $X(s) \leq M \ \forall s \in S$)}
\]

\[
= M \sum_{s \in S} p(s) \cdot Y(s)
\]

\[
= M \cdot E(Y) \quad \text{(by definition)}
\]
Problem 10.
Let $S$ be the set of permutations of 10 objects.

10(a) [2 points]
How many elements of $S$ leave the third object fixed?

Ans. Permutations that fix the third object correspond precisely to permutations of the $n-1 = 9$ objects numbered 1, 2, 4, 5, ..., 10. Thus there are $9!$ of them.

10(b) [3 points]
Consider the experiment of choosing an element of $S$ randomly with equal probabilities. Let $R$ be the random variable defined by $R(\sigma) = \{|j : \sigma(j) = j\}|$ where $\sigma$ is a permutation in $S$, so $R(\sigma)$ is the number of fixed points of $\sigma$. Compute the expected value of $R$.

Ans. Let $n = 10$. For $j = 1$ to $n$ define an indicator random variable $R_j$ by $R_j(\sigma) = 1$ if $\sigma$ fixes $j$ and $R_j(\sigma) = 0$ otherwise. Then the number of fixed points is $R = R_1 + R_2 + \cdots + R_n$. Since expectation is linear,

$$E(R) = E(R_1) + E(R_2) + \cdots + E(R_n).$$

But precisely $(n-1)!$ of the permutations of $n$ objects fix the first object, as in part (a), so

$$E(R_1) = \frac{(n-1)!}{n!} = \frac{1}{n}.$$

Since there was nothing special about the first object, $E(R_j) = 1/n$ for each $j$. Thus $E(R) = n \cdot 1/n = 1$. 

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Problem 11.
Suppose that two fair 8-sided dice are rolled.

11(a) [2 points]
What is the expected value of the sum of the numbers that come up?

Ans. Since expected value is linear, the expected value of the sum is equal to the sum of the expected values. For each die, each of the outcomes 1 through 8 occurs with probability 1/8, so the expected value is \((1/8)(1 + 2 + 3 + \cdots + 8) = 9/2\). Therefore the answer is 9.

10(b) [3 points]
What is the variance of the sum of the numbers that come up?

Ans. Since variance is linear for independent random variables, and these random variables are independent, the variance of the sum is equal to the sum of the variances. For each die, \(V(X) = E(X^2) - E(X)^2 = (1/8)(12 + 22 + 32 + \cdots + 82) - (9/2)^2 = (51/2) - (81/4) = 21/4\). Therefore the answer is 21/2.
Problem 12.
Is the following pair of graphs isomorphic? If yes, give an explicit isomorphism; if no, explain why they are not isomorphic.

Isomorphic: Yes/No

Ans. Yes, $G$ and $H$ are isomorphic.
Justification:

Ans. Bijection follows: $u_1 \leftrightarrow v_2, u_2 \leftrightarrow v_1, u_3 \leftrightarrow v_3, u_4 \leftrightarrow v_4, u_5 \leftrightarrow v_8, u_6 \leftrightarrow v_7, u_7 \leftrightarrow v_5, u_8 \leftrightarrow v_6,$
Problem 13.

13(a) [3 points]
Prove that every connected undirected simple graph on $n$ vertices has at least $n - 1$ edges.

\textit{Ans.} Proof by induction on $n$. Basis: $n = 1$. A simple graph with 1 vertex has $1 - 1 = 0$ edges. Inductive step: Assume the inductive hypothesis: every connected simple graph on $k$ vertices has at least $k - 1$ edges, for some $k \geq 2$. Let $G$ be a connected simple graph with $k + 1$ vertices: we must show that $G$ has at least $k$ edges.

Since $G$ is connected, every vertex of $G$ must have degree 1 or greater. Case 1: if every vertex of $G$ has degree 2 or greater, then the sum of the degrees is at least $2(k + 1)$. Therefore, by the handshake theorem, the number of edges is at least $k + 1$, which is greater than $k$.

Case 2: suppose $G$ has at least one vertex of degree 1. Call it $v$. Remove $v$ and its adjacent edge from $G$. Since $\text{deg}(v) = 1$, this does not disconnect $G$. Thus $G - v$ is still connected, and it has $k$ vertices, so by the inductive hypothesis, $G - v$ has at least $k - 1$ edges. Since $G$ has one more edge than $G - v$, $G$ has at least $k$ edges, and the proof is complete.

13(b) [2 points]
Prove that every undirected simple graph on $n$ vertices with $k$ connected components has at least $n - k$ edges.

\textit{Ans.} Generalize your proof for part a!
Problem 14.

14(a) [1 points]
Complete the following definition: “A tree is . . . ”.

Ans. A tree is a connected undirected graph with no simple circuits.

14(b) [4 points]
Prove that a simple undirected graph is a tree if and only if it contains no simple circuits and the addition of an edge connecting two nonadjacent vertices produces a new graph that has exactly one simple circuit (where circuits that contain the same edges are not considered different).

Ans. First assume that $G$ is a tree. We must show that $G$ contains no simple circuits (which is immediate by definition) and that the addition of an edge connecting two nonadjacent vertices produces a graph that has exactly one simple circuit. Clearly the addition of such an edge $e = (u, v)$ results in a graph with a simple circuit, namely $u, e, v, P, u$, where $P$ is the unique simple path joining $v$ to $u$ in $G$. Since $P$ is unique, this is the only simple circuit that can be formed. (You may need to prove property: An undirected graph is a tree if and only if there is a unique simple path between any two of its vertices.)

To prove the converse, suppose that $G$ satisfies the given conditions; we want to prove that $G$ is a tree; i.e., that $G$ is connected (since one of the conditions is already that $G$ has no simple circuits). Suppose, to the contrary, that $G$ is not connected. Then there exists two vertices $u$ and $v$ that lie in different connected components of $G$. Then edge $(u, v)$ can be added to $G$ without the formation of any simple circuits, contradicting the assumed condition. Therefore $G$ is indeed a tree.
Problem 15.
Let $G$ be a nonempty finite simple undirected graph.
Prove: if every vertex of $G$ has even degree, then $G$ contains a simple circuit.

Ans. Let $P = u_1, u_2, \ldots, u_k$, where $u_i$ are vertices in $G$, be a simple path of maximal length in $G$. Since $\deg(u_1) \geq 2$, there is at least one other edge $(v, u_1)$ in $G$ having $u_1$ as a vertex. If $v$ is one of the vertices of the path $P$, then the subpath from $u_1$ to $v$, concatenated with the edge $(v, u_1)$, forms a simple circuit in $G$. If $v$ is not a vertex of path $P$, then the path $v, u_1, u_2, \ldots, u_k$ is a simple path whose length is greater than the length of $P$, contradicting the assumption that the length of $P$ is maximal.
Problem 16.

Find a recurrence to describe the number of ways to completely cover a $2 \times n$ checkerboard with $1 \times 2$ dominoes.

Ans. Let $a_n$ be the number of coverings. Consider separately the covering where the position in the top right corner of the checkerboard is covered by a domino positioned horizontally and where it is covered by a domino positioned vertically.

If the rightmost domino is positioned vertically, then we have a covering of the leftmost $n - 1$ columns, and this can be done in $a_{n-1}$ ways. If the rightmost domino is positioned horizontally, then there must be another domino directly beneath it, and these together cover the last two columns. The first $n - 2$ columns therefore will need to contain a covering by dominoes, and this can be done in $a_{n-2}$ ways. Thus we obtain the recurrence $a_n = a_{n-1} + a_{n-2}$, the Fibonacci recurrence.